Introduction to Physics for Scientists and Engineers

Third Edition

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9.3 ANGULAR SPEED AND VELOCITY

The angular quantities velocity and acceleration are defined in a way much like their linear counterparts. For example, linear velocity was defined as

$$\mathbf{v} = \frac{\Delta \mathbf{s}}{\Delta t}$$

where $\Delta \mathbf{s}$ is the vector displacement in time Δt. Similarly, if an angle $\Delta \boldsymbol{\theta}$ is swept out in time Δt, the angular velocity ω (omega) is defined to be

Definition of angular velocity

$$\omega = \frac{\Delta \boldsymbol{\theta}}{\Delta t} \qquad \text{or} \qquad \omega = \frac{d\boldsymbol{\theta}}{dt} \tag{9.1a}$$

Notice that the units of ω are those of $(\text{time})^{-1}$. However, to indicate the angular measure being used, we shall state ω to be in radians per second, revolutions per minute, or some other similar form. Notice that ω is defined in terms of an *incremental* rotation, and so it obeys the usual vector addition laws.

If the value of $\Delta t \to 0$, then *9.1a* is the definition of instantaneous angular velocity. For larger time intervals, ω becomes an average over that time. Because the rotation vectors we consider in this chapter are all parallel or antiparallel to the same direction, we are not concerned with the vector nature of the quantity and merely use the magnitudes of the quantities to give

$$\Delta\theta = \omega\,\Delta t \tag{9.1b}$$

This equation will then define the angular speed ω. As we shall see, the distinction between speed and velocity is less often made in angular than in linear motion.

9.4 ANGULAR ACCELERATION

Like angular velocity, angular acceleration is defined in analogy to the similar linear motion quantity. In the case of linear motion we had

$$\mathbf{a} = \frac{\mathbf{v} - \mathbf{v}_0}{t}$$

where t is the time taken for the velocity to change from $\mathbf{v}_0$ to $\mathbf{v}$. The angular acceleration $\boldsymbol{\alpha}$ is defined by

Definition of angular acceleration

$$\boldsymbol{\alpha} = \frac{\omega - \omega_0}{t} \qquad \text{or} \qquad \boldsymbol{\alpha} = \frac{d\omega}{dt} \tag{9.2}$$

where t is the time taken for the angular velocity to change from ω_0 to ω. Since the units of ω are 1/time, the units of α must be $1/(\text{time})^2$. But we shall state α as rad/s^2, etc., in order to indicate the angular measure as well.

You will recall from the discussion of linear acceleration that a distinction is made between instantaneous and average acceleration. Here, too, the same distinction can be made, depending upon whether or not $t \to 0$. However, since we shall usually be concerned with constant acceleration, this distinction will not often be of importance for us. One must remember, though, that the equations we shall next obtain are valid only for the case of α constant.

When the angular acceleration is constant, the average angular speed

CALCULUS

With Analytic Geometry

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In Chapter 2 the slope m of the tangent line l at P was defined as the limiting value of m_{PQ} as Q approaches P (see (ii) of Figure 3.1). If f is continuous, then we can make Q approach P by letting h approach 0. It is natural, therefore, to define m as follows.

(3.1) **Definition**

If a function f is defined on an open interval containing a, then the slope m of the tangent line to the graph of f at the point $P(a, f(a))$ is given by

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided the limit exists.

If a tangent line is vertical, its slope is undefined and the limit in (3.1) does not exist. Vertical tangent lines will be studied in the next chapter.

Example 1 If $f(x) = x^2$, find the slope of the tangent line to the graph of f at the point $P(a, a^2)$.

Solution This problem is the same as the one stated for equations in Example 1 of the first section in Chapter 2. Using the quotient in Definition (3.1),

$$\begin{aligned}\frac{f(a+h) - f(a)}{h} &= \frac{(a+h)^2 - a^2}{h} \\ &= \frac{a^2 + 2ah + h^2 - a^2}{h} \\ &= \frac{2ah + h^2}{h} \\ &= 2a + h.\end{aligned}$$

The slope m of the tangent line is, therefore,

$$m = \lim_{h \to 0} (2a + h) = 2a.$$

One of the main reasons for the invention of calculus was the need for a way to study the behavior of objects in motion. Let us consider the problem of arriving at a satisfactory definition for the velocity, or speed, of an object at a given instant. We shall assume, for simplicity, that the object is moving on a straight line. Motion on a straight line is called **rectilinear motion.** It is easy to define the **average velocity** v during an interval of time. We merely use the formula

(3.2) $$v = \frac{d}{t}$$

12.8 TAYLOR AND MACLAURIN SERIES

Suppose a function f is represented by a power series in $x - c$, such that

(12.36)
$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$$
$$= a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots + a_n(x-c)^n + \cdots$$

where the domain of f is an open interval containing c. As in the preceding section, power series representations may be found for $f'(x)$, $f''(x)$, ..., by differentiating the terms of the series in (12.36). Thus

$$f'(x) = \sum_{n=1}^{\infty} na_n(x-c)^{n-1}$$
$$= a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + 4a_4(x-c)^3 + \cdots$$
$$f''(x) = \sum_{n=2}^{\infty} n(n-1)a_n(x-c)^{n-2}$$
$$= 2a_2 + (3\cdot 2)a_3(x-c) + (4\cdot 3)a_4(x-c)^2 + \cdots$$
$$f'''(x) = \sum_{n=3}^{\infty} n(n-1)(n-2)a_n(x-c)^{n-3}$$
$$= (3\cdot 2)a_3 + (4\cdot 3\cdot 2)a_4(x-c) + \cdots$$

and, for every positive integer k,

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(x-c)^{n-k}.$$

.., each series obtained by differentiation has the same radius of convergence as the original series. Substituting c for x in each of these series representations, we obtain

$$f(c) = a_0, \quad f'(c) = a_1, \quad f''(c) = 2a_2, \quad f'''(c) = (3\cdot 2)a_3$$

and, for every positive integer n,

$$f^{(n)}(c) = n!\,a_n, \quad \text{or} \quad a_n = \frac{f^{(n)}(c)}{n!}.$$

We have proved the following result.

(12.37) **Theorem**

If f is a function such that

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$$

for all x in an open interval containing c, then

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

The series which appears in the conclusion of the last theorem is called the **Taylor series for $f(x)$ at c**. The special case $c = 0$ is very important and hence is stated separately as a corollary.

(12.38) **Corollary**

> If f is a function such that $f(x) = \sum a_n x^n$ for all x in an open interval $(-r, r)$, then
>
> $$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots.$$

The series appearing in this Corollary is called the **Maclaurin series for $f(x)$**. Each example in the preceding section involves a Maclaurin series.

Theorem (12.37) states that *if* a function f is represented by a power series in $x - c$, then the power series *must* be the Taylor series. However, the theorem does not state conditions which guarantee that a power series representation actually exists. We shall now obtain such conditions. Let us begin by noting that the $(n + 1)$st partial sum of the Taylor series stated in (12.37) is the nth-degree Taylor Polynomial $P_n(x)$ of f at c (see (11.13)). Moreover, by (11.15),

(12.39) $$P_n(x) = f(x) - R_n(x)$$

where

(12.40) $$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x - c)^{n+1}$$

some number z between c and x. In the next theorem we use $R_n(x)$ to specify sufficient conditions for the existence of a power series representation of $f(x)$.

(12.41) **Theorem**

> If a function f has derivatives of all orders throughout an interval containing c, and if
>
> $$\lim_{n\to\infty} R_n(x) = 0$$
>
> for every x in that interval, then $f(x)$ is represented by the Taylor series for $f(x)$ at c.

Proof. The polynomial $P_n(x)$ is a general term for the sequence of partial sums of the Taylor series for $f(x)$ at c. Moreover, from (12.39),

$$\lim_{n\to\infty} P_n(x) = f(x) - \lim_{n\to\infty} R_n(x) = f(x).$$

Hence the sequence of partial sums converges to $f(x)$. This proves the theorem.

NUMERICAL SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS IN SCIENCE AND ENGINEERING

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Thus two possible approximations to the first derivative of u at x_r are given by (2.1.4):

$$u_x\Big|_r \approx \frac{u(x_r+h)-u(x_r)}{h} \equiv \frac{u_{r+1}-u_r}{h} \tag{2.1.5a}$$

$$u_x\Big|_r \approx \frac{u(x_r)-u(x_r-h)}{h} \equiv \frac{u_r-u_{r-1}}{h}. \tag{2.1.5b}$$

Because the series has been arbitrarily truncated, there is clearly an error, E_r say, associated with this approximation. This error can be characterized by the first and largest term of the truncated series, which yields

$$E_r = \pm\frac{h}{2}u_{xx}\Big|_\xi = O(h), \qquad \begin{matrix} x_r \leqslant \xi \leqslant x_r+h \\ x_r-h \leqslant \xi \leqslant x_r. \end{matrix} \tag{2.1.6}$$

We say that this error is of order h, $O(h)$. The $O(h)$ error is in absolute value smaller than Ah (A a constant) for sufficiently small h.

If we add (2.1.4a) and (2.1.4b) and solve for $u_x|_r$, there results

$$u_x\Big|_r = \frac{u_{r+1}-u_{r-1}}{2h}, \tag{2.1.7}$$

with the first tr ng

$$-\frac{h^2}{6}u_{xxx}\Big|_\xi, \qquad x_{r-1} \leqslant \xi \leqslant x_{r+1}.$$

(2.1.7) is $O(h^2)$. Subtracting (2.1.4b) from (2.1.4a) and solving for $u_{xx}|_r$, we obtain

$$u_{xx}\Big|_r = \frac{u_{r-1}-2u_r+u_{r-1}}{h^2}, \tag{2.1.8}$$

with the truncated first term as

$$\frac{-h^2}{12}u_{4x}\Big|_\xi, \qquad x_{r-1} \leqslant \xi \leqslant x_{r+1}.$$

Thus (2.1.8) is $O(h^2)$. Although we could continue in this fashion to develop more complex formulas, the manipulations become excessive. Instead, we turn to the use of operator notation.